A Simple Proof of the FWL Theorem

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Abstract: The author presents a simple proof of a property of the method of least squares variously known as the FWL, the Frisch-Waugh-Lovell, the Frisch-Waugh, or the decomposition theorem.

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Ragnar Frisch and F. V. Waugh (1933) demonstrated a remarkable property of the method of least squares in a paper published in the first volume of *Econometrica*. Suppose one is fitting by least squares the variable Y_t on a set of k' explanatory variables, plus a linear time trend, $t = 1, 2, \ldots$

$$Y_t = b_0 + b_1 X_{1t} + b_2 X_{2t} + \cdots + b_{k'} X_{k't} + dt + e_t. \tag{1}$$

As an alternative to the direct application of least squares, they considered the following two-step trend-removal procedure:

Step 1. Detrend all the X_{it} and Y_t by first regressing each on the time variable,

$$X_{it} = c_{i0} + c_{i1} t + e^x_{it}, \quad i = 1, \ldots, k', \quad \text{and} \tag{2}$$

$$Y_t = c_0 + c_1 t + e^y_t, \tag{3}$$

and using the residuals from these least-squares regressions to calculate the detrended variables,

$$X^*_{it} = \bar{X}_i + e^x_{it}, \quad i = 1, \ldots, k', \quad \text{and} \tag{4}$$

$$Y^*_t = \bar{Y} + e^y_t. \tag{5}$$

Step 2: Run the detrended regression,

$$Y^*_t = b^*_0 + b^*_1 X^*_{1t} + b^*_2 X^*_{2t} + \cdots + b^*_{k'} X^*_{k't} + e^*_t. \tag{6}$$

Frisch and Waugh proved a surprising proposition:

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Exactly the same coefficients are obtained with regression (6), based on detrended variables, as with regression (1), which includes trend as an explanatory variable that is,

$$b_i^* = b_i, \quad \text{for } i = 1, \ldots, k'.$$

It is important to note that the fact that the least-squares regression coefficients b_i and b_i^* are identical means that neither is superior to the other as an estimator of the unknown parameters β_i of the underlying stochastic process that may be generating the data. It is also true that the residuals from regressions (1) and (6) are also identical, $e_t = e_t^*$, which obviously means that examining either set of residuals will convey precisely the same information about the properties of the unobservable stochastic disturbances ε_t.

Lovell (1963) generalized Frisch and Waugh's (1933) result by showing that the same regression coefficients will be obtained not just with a trend variable but with seasonal variables or, indeed, any nonempty subset of the explanatory variables in a regression. This result is variously known as the FWL, the Frisch-Waugh-Lovell, the Frisch-Waugh, or the decomposition theorem.

The FWL Theorem

Suppose we partition the explanatory variables of a k variable multiple regression into any two nonempty sets, one consisting of k' variables X_{it} on which our attention is primarily focused and the other a set of $k'' = k - k'$ auxiliary variables D_{it}:

$$\begin{aligned} Y_t = b_1 X_{1t} + b_2 X_{2t} + \cdots + b_{k'} X_{k't} + d_1 D_{1t} + d_2 D_{2t} \\ + \cdots + d_{k''} D_{k''t} + e_t. \end{aligned} \tag{7}$$

To ease notation, follow the standard convention of subsuming the intercept in the other explanatory variables by setting all values of an additional explanatory variable identically equal to 1. Now consider the alternative least-squares regression equation:

$$Y_t^* = b_1^* X_{1t}^* + b_2^* X_{2t}^* + \cdots + b_{k'}^* X_{k't}^* + e_t^*. \tag{8}$$

Here, the Y_t^* and X_{it}^* are cleansed values of the dependent variable and the focus subset of the explanatory variables.

$$\begin{aligned} Y_t^* &= \bar{Y} + e_t^y \qquad \text{and} \\ X_{it}^* &= \bar{X}_i + e_{it}^x, \quad i = 1 \,..\, k', \end{aligned} \tag{9}$$

where e_t^y and the e_{it}^x are the least-squares residuals obtained from the auxiliary regressions

$$Y_t = a_{y1} D_{1t} + \cdots + a_{yk''} D_{k''t} + e_t^y \tag{10}$$

$$X_{it} = a_{i1} D_{1t} + \cdots + a_{ik''} D_{k''t} + e_{it}^x, \quad i = 1..k'. \tag{11}$$

Then,

$$b_i^* = b_i \qquad \text{for } i = 1, \,..\,, k' \qquad \text{and} \tag{12}$$

$$e_t^* = e_t. \tag{13}$$

Frisch and Waugh had employed Cramer's rule in proving their trend theorem whereas Lovell (1963, 1007–8) used matrix algebra in establishing the more general FWL theorem. Davidson and MacKinnon (1999, 62–69) presented both a geometric demonstration and a matrix proof of the result in their econometrics textbook; Greene (2003, 26–27) and Johnston and Dinardo (1997, 101–3) employed matrix algebra in their texts.

In this article, I use simple algebra to show how the FWL theorem can be easily derived from two well-known numerical properties of the method of least squares:

Property 1. The residuals from a least-squares regression are uncorrelated with the explanatory variables.

Property 2. The coefficients of a subset of the explanatory variables in a regression equation will be zero if those variables are uncorrelated with both the dependent variable and the other explanatory variables.[1]

Proof:

Substituting equation (10) into equation (7) yields

$$\begin{aligned} e_t^y &= b_1 e_{1t}^x + \cdots + b_{k'} e_{k't}^x + (b_1 a_{11} + \cdots + b_{k'} a_{k'1} + d_1 - a_{y1}) D_{1t} + \cdots \\ &\quad + (b_1 a_{1k''} + \cdots + b_{k'} a_{k'k''} + d_{k''} - a_{yk''}) D_{k''t} + e_t. \end{aligned} \tag{14}$$

Because auxiliary regressions (10) and (11) are fitted by the method of least squares, Property 1 implies that the residuals e_{it}^x and e_t^y from those regressions are uncorrelated with the D_{it} explanatory variables. Therefore, all the regression coefficients of the D_{it} in equation (14) are zero, thanks to Property 2, which means that precisely the same b_i are obtained when the D_{it} are dropped from the regression; that is,

$$e_t^y = b_1 e_{1t}^x + b_2 e_{2t}^x + \cdots + b_{k'} e_{k't}^x + e_t. \tag{15}$$

Adding the identity $\bar{Y} = b_1 \bar{X}_1 + b_2 \bar{X}_2 + \cdots b_{k'} \bar{X}_{k'}$ to equation (15) yields

$$\bar{Y} + e_t^y = b\left(\bar{X}_1 + e_{1t}^x\right) + b_2\left(\bar{X}_2 + e_{2t}^x\right) + \cdots + b_{k'}\left(\bar{X}_{k'} + e_{k't}^x\right) + e_t, \tag{16}$$

which by equation (9) is equation (8), thus establishing that the least-square coefficients b_i^* of equation (8) are identical to the b_i of equation (7) and that $e_t^* = e_t$.

COMMENTS

1. There are $n - k < n - k'$ degrees of freedom in regressions (8) and (15) as well as (7). Therefore, execution of either regression (8) or (15) with a standard least-squares regression computer program neglecting this complication will yield too small a value for the standard error of the estimate, $\bar{S}_e$, and exaggerated t and p values for the regression coefficients (Lovell 1963, 1002-3).

2. Because the least-squares residuals calculated with regressions (7) and (8) are identical, precisely the same Durbin-Watson statistics will be generated.
3. The application of Aitkens Generalized Least Squares to regression equations (8) or (15) will result in less-efficient estimates than its direct application to regression (7) (Lovell 1963, 1004).
4. Precisely the same regression coefficients but different residuals are generated when Y_t instead of Y_t^* is used as the dependent variable in regression (8) (Lovell 1963, 1001).

NOTE

1. This is easily seen in the simplest case of only two explanatory variables: The first multiple-regression coefficient, given the presence of x_2, is

$$b_{1.2} = \left(\sum yx_1 \cdot \sum x_2^2 - \sum yx_2 \cdot \sum x_1x_2\right) \Big/ \left[\sum x_1^2 \cdot \sum x_2^2 - \left(\sum x_1x_2\right)^2\right],$$

which reduces to

$$b_1 = \sum yx_1 / \sum x_1{}^2 \quad \text{if} \quad \sum x_1x_2 = ns_{x_1}s_{x_2}r_{12} = 0;$$

$$\text{if in addition} \quad r_{x_1y} = 0, \text{ then } b_1 = 0.$$

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